Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 270 (2004) 470–472

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Letter to the Editor

Oscillators with a generalized power-form elastic term

V.N. Pilipchuk

Department of Mechanical Engineering, Wayne State University, Detroit, MI 48202, USA

Received 30 January 2003; accepted 22 April 2003

In recent works [1–3], the authors brought attention to non-linear oscillators with non-polynomial characteristics,

$$\ddot{x} + x^{\alpha} = 0, \tag{1}$$

where $\alpha = (2k+1)/(2n+1)$. In particular, it was shown how to apply the harmonic balance method under arbitrary non-negative integers k and n [3]. Oddness of both numerator and denominator of the exponent is important. If one of the parts of the ratio is even then system (1) is not an oscillator. Obviously, the numbers α represent a subset of the positive rational numbers, but do not include all of them. Note that the particular case $n = 0$ has been investigated in the literature for a long time. For example when investigating degeneralized cases of stability problems, A.M. Lyapunov introduced special periodic functions in order to represent the solutions [4]. Another version of the special functions was used in Ref. [5]. In order to deal with the class of elementary functions and get more physical insight, asymptotic approaches are being developed [6,7], see also references therein. The physical nature of such asymptotics is due to the fact that the corresponding potential $x^{2k+2}/(2k+2)$ approaches the quare-well form as $k \to \infty$. As a result, the limit system oscillates in the interval $-1 \leqslant x \leqslant 1$ between the two absolutely rigid and perfectly elastic barriers located at the ends of the interval.

As was shown in Ref. [6], for arbitrary positive odd integer $\alpha = 2k+1$, a periodic solution of oscillator (1) can be represented in the form

$$x = X(\tau), \quad \tau = \tau(t/a), \tag{2}$$

where $X(-\tau) = -X(\tau)$, and

$$\tau(\xi) = \frac{2}{\pi} \arcsin \sin \frac{\pi \xi}{2}, \quad \tau(\xi) = \tau(\xi + 4) \tag{3}$$

is the periodic saw-tooth function of the period $T = 4a$. Both the function $X(\tau)$ and the parameter a are determined by successive approximations in the form of series as those reproduced below in the modified form.

E-mail address: pylyp@me1.eng.wayne.edu (V.N. Pilipchuk).

doi:10.1016/S0022-460X(03)00531-5

In this paper, we generalize solution (2) as

$$x = \mathrm{sgn}(\tau)X(|\tau|). \tag{4}$$

Such a modification enables one to consider a more general class of oscillators:

$$\ddot{x} + \mathrm{sgn}(x)|x|^{\alpha} = 0. \tag{5}$$

In the case $\alpha = (2k+1)/(2n+1)$, oscillator (5) is absolutely identical to the original one, Eq. (1), as well as solution (4) is identical to solution (2) due to its oddness with respect to τ. But the generalized form of Eq. (5) allows the exponent α to continuously take any non-negative real value, such as odd, even, rational or irrational: $0 \leqslant \alpha < \infty$. For example, $\ddot{x} + \mathrm{sign}(x)|x|^{3/2} = 0$ is an oscillator with the odd characteristic, whereas $\ddot{x} + x^{3/2} = 0$ is not an oscillator at all. The form of solution (4) is dictated by the symmetries of system (5). Note that the characteristic of oscillator (5) is given by the potential $V(x) = -|x|^{\alpha+1}/(\alpha+1)$ such that the differential equation of motion (5) can be represented in the standard form $\ddot{x} = V'(x)$ regardless of the presence of the non-smooth function $|x|$.

By applying transformation (4) to the solution $X(\tau)$ derived in Ref. [6], one obtains

$$x = A\,\mathrm{sgn}(\tau)\left[|\tau| - \frac{|\tau|^{\alpha+2}}{\alpha+2} + \frac{\alpha}{2(\alpha+2)}\left(\frac{|\tau|^{2\alpha+3}}{2\alpha+3} - \frac{|\tau|^{\alpha+2}}{\alpha+2}\right) + \cdots\right], \tag{6}$$

$$a^2 = \frac{\alpha+1}{A^{\alpha-1}}\left\{1 + \frac{\alpha}{2(\alpha+2)} + \frac{\alpha^2}{4(\alpha+2)^2}\left[1 + \frac{\alpha+2}{\alpha(2\alpha+3)}\right] + \cdots\right\}. \tag{7}$$

where A is an arbitrary parameter characterizing the amplitude, whereas another arbitrary parameter t_0 can be introduced by substituting $t \to t + t_0$.

For any $|\tau| \leqslant 1$ and α, convergence of series (6) and (7) is not less rapid than that of the geometric series with the common ratio $\frac{1}{2}$. However, due to the structure of the solution, best results are achieved for large exponents α, when a quasi-harmonic approach may give a significant

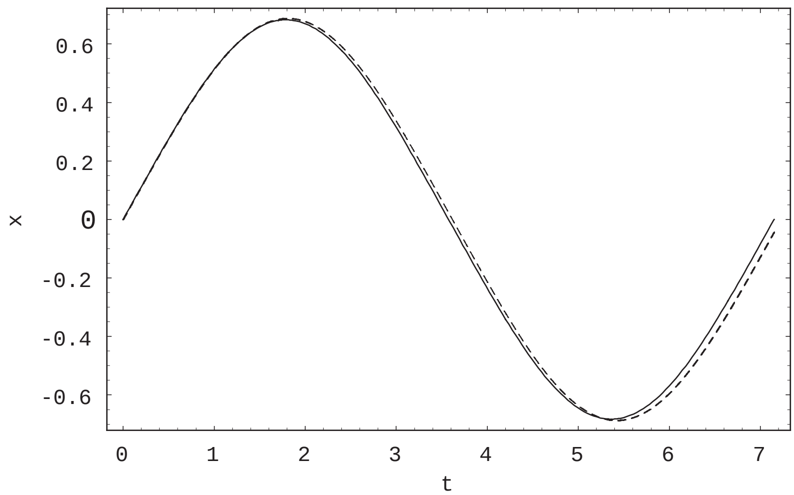

Fig. 1. Analytical and numerical solutions of the modified oscillator shown by continuous and dashed lines, respectively, $\alpha = \frac{3}{2}$.

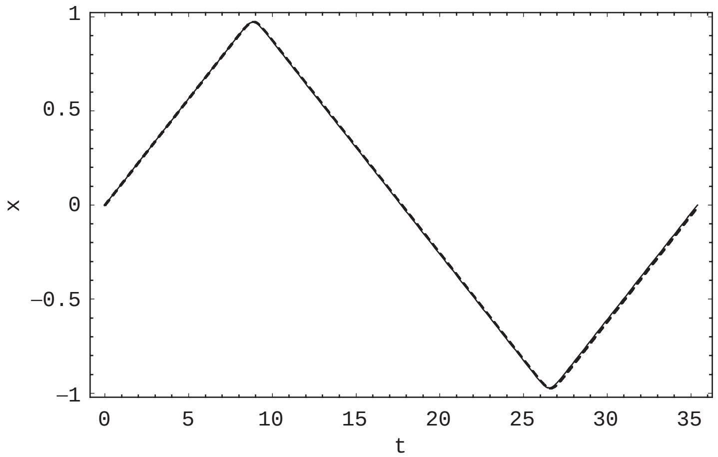

Fig. 2. Approaching the saw-tooth temporal mode shape with a good match of analytical and numerical solutions shown by continuous and dashed lines, respectively, $\alpha = \sqrt{2003}$.

error due to a strongly unharmonic temporal mode shape of the vibration. Expressions (6) and (7) show explicitly three terms of each of the series that were taken for calculations.

For example, Figs. 1 and 2 show solution (6) in comparison with numerical solution for two different exponents $\alpha = \frac{3}{2}$ and $\sqrt{2003}$, respectively, and the same parameter $A = 1$. As the figures show the analytical and numerical solutions are matching better as the exponent α increases.

References

[1] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, *Journal of Sound and Vibration* 246 (2001) 375–378.
[2] R.E. Mickens, Analysis of non-linear oscillators having non-polynomial elastic terms, *Journal of Sound and Vibration* 255 (2002) 789–792.
[3] H. Hu, Z.-G. Xiong, Oscillations in an $x^{(2m+2)/(2n+1)}$ potential, *Journal of Sound and Vibration* 259 (2003) 977–980.
[4] G.V. Kamenkov, *Stability and Vibrations of Nonlinear Systems*, Vol. 1, Nauka, Moscow, 1972 (in Russian).
[5] R.M. Rosenberg, The Ateb(h)-functions and their properties, *Quarterly of Applied Mathematics* 21 (1963) 37–47.
[6] V.N. Pilipchuk, Analytical study of vibrating systems with strong non-linearities by employing saw-tooth time transformations, *Journal of Sound and Vibration* 192 (1996) 43–64.
[7] I.V. Andrianov, Asymptotics of nonlinear dynamical systems with a high degree of nonlinearity, *Doklady Mathematics* 66 (2002) 270–273.